On the isomorphism problem for relatively hyperbolic groups

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(joint work with François Dahmani)

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The isomorphism problem

Given two finite presentations

$$\langle x_1, \ldots, x_n \mid r_1, \ldots r_m \rangle ; \langle y_1, \ldots, y_p \mid s_1, \ldots, s_q \rangle$$

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In general no (Adian, Rabin late 1950s.)

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That being said this is still a reasonnable question to ask if the presentation are known to lie a restricted class of groups. For example if both presentations are known to be of abelian groups, then we can straightforwardly decide if these two presentations give isomorphic groups.

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- Dahamni and Groves solve the isomorphism problem for toral relatively hyperbolic groups (2008), this class includes torsion-free hyperbolic groups and limit groups.
- Dahmani and Guirardel solve the isomorphism problem for all hyperbolic groups; even those with torsion (2011).

A new result

Theorem (Dahmani-T)

There is an algorithm which given two presentations

$$\langle x_1, \ldots, x_n \mid r_1, \ldots r_m \rangle; \langle y_1, \ldots, y_p \mid s_1, \ldots, s_q \rangle$$

of torsion-free groups that are known to be hyperbolic relative to nilpotent groups, will decide if they are isomorphic.

Some terminology: Peripheral structures and relative hyperbolicity

Let G be a group, a *peripheral structure* $\mathcal{P}$ *on* G is a (possibly empty) union of finitely many conjugacy classes:

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$$(\mathbb{Z}^2 * \mathbb{Z}^2, \mathcal{P})$$

where $\mathcal{P}$ is the set of maximal non-cyclic abelian subgroups is a relatively hyperbolic group.

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We also say that G is *hyperbolic relative to* $\mathcal{P}$.

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An *elementary splitting* of $(G, \mathcal{P})$ is an (essential) decomposition of G as the fundamental group of a graph of groups such that:

- Edge groups are parabolic, virtually cyclic, or finite,
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We say that a relatively hyperbolic group $(G, \mathcal{P})$ is *rigid* if it is not virtually cyclic and it admits no elementary splittings.

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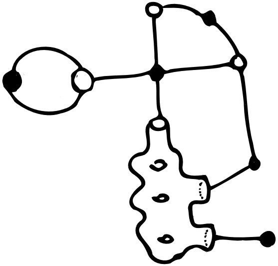
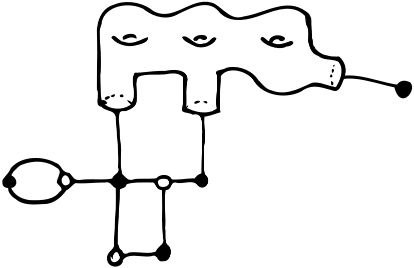


JSJ(Γ)



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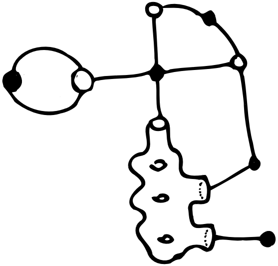
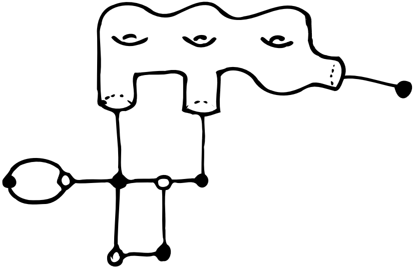


$\mathrm{JSJ}(\Gamma)$



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So now we check if the graphs of groups "look" the same. I.e. if the underlying graphs are isomorphic.

The general strategy for the isomorphism problem

- Step 2: The black vertex groups are either *periperal or virtually cyclic*. We enlarge the peripheral structures $\mathcal{P} \subset \hat{\mathcal{P}}, \mathcal{Q} \subset \hat{\mathcal{Q}}$ so that the black vertex groups are peripheral. The white vertex groups are either *QH (or surface type)* or (w.r.t. the natural induced rel. hyp. structure) *rigid*. We now solve the isomorphism problem for the vertex groups*.

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- Step 3: If the previous steps went though, see if the graphs of groups assemble to give isomorphic groups.

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be rigid hyperbolic groups. The theory of actions on $\mathbb{R}$-trees tells us that either:

- There are monomorphisms $\Gamma \hookrightarrow \Delta$ and $\Delta \hookrightarrow \Gamma$, which by co-Hopficity, implies that $\Delta \approx \Gamma$; or

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- There are monomorphisms $\Gamma \hookrightarrow \Delta$ and $\Delta \hookrightarrow \Gamma$, which by co-Hopficity, implies that $\Delta \approx \Gamma$; or
- W.l.o.g. there is some finite set $F \subset \Gamma$ such that for every $f \in \mathrm{Hom}(\Gamma, \Delta)$ $f|_F$ is not injective. We call such a set an *obstruction* from Γ to Δ.

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We now have two processes.

Process 1: Enumerate via Tietze transformations all presentations isomorphic to $\langle x_1, \ldots, x_n \mid r_1, \ldots r_m \rangle$. If our presentation $\langle y_1, \ldots, y_p \mid s_1, \ldots, s_q \rangle$ of Δ appear stop and output "$\Gamma \approx \Delta$".

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Process 2: Look for obstructions from Δ to Γ and vice versa. Take $F_1 \subset F_2 \subset \ldots$, $E_1 \subset E_2 \subset \ldots$ finite exhaustions (group elements represented as words) of $\Gamma \setminus \{1\}, \Delta \setminus \{1\}$ and check the truth of the first order sentences*

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$$\Gamma \models \exists y_1, \ldots, y_p \left(\left(\bigwedge_{i=1}^{q} s_i(y_1, \ldots, y_p) = 1 \right) \wedge \left(\bigwedge_{f \in E_k} f(y_1, \ldots, y_p) \neq 1 \right) \right)$$

$$\Delta \models \exists x_1, \ldots, x_n \left(\left(\bigwedge_{i=1}^{m} r_i(x_1, \ldots, x_n) = 1 \right) \wedge \left(\bigwedge_{e \in F_k} e(x_1, \ldots, x_n) \neq 1 \right) \right)$$

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Dahmani and Guirardel then showed that this is decidable for virtually free groups and used this for the case of arbitrary hyperbolic groups.

The isomorphism problem: an existential crisis

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To make things worse Step 1 in the previous methods (compute the JSJ) also heavily makes use of equationnal methods.

Dehn fillings.

Let $(G, \mathcal{P})$ and $(H, \mathcal{Q})$ relatively hyperbolic groups with $\mathcal{P} = [P_1] \cup \cdots \cup [P_s], \mathcal{Q} = [Q_1] \cup \cdots \cup [Q_s]$. With $\mathcal{Q}, \mathcal{P}$ collections of *residually finite groups**.

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$$K_n = \langle\langle P_1^{(n)}, \ldots, P_s^{(n)} \rangle\rangle \lhd G \text{ where } P_i^{(n)} = \bigcap_{[P_i:H] \leq n} H$$

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and define $L_n \lhd H$ similarly.
We call G/K_n the n^{th} *characteristic Dehn filling.* For $n >> 0$ G/K_n is hyperbolic relative to

$$\mathcal{P}_n = \bigcup_{i=1}^{s} [P_i/(P_i \cap K_n)] \text{ and } P_i/K_n \approx P_i/P_i^{(n)}.$$

This is due to Osin and Groves-Manning.

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$$\begin{tikzcd}
(G, \mathcal{P}) \arrow[d, two heads] & (H, \mathcal{Q}) \arrow[d, two heads] \\
(G/K_m, \mathcal{P}_m) \arrow[r, "\alpha_m", "\approx"'] \arrow[d, two heads] & (H/L_m, \mathcal{Q}_m) \arrow[d, two heads] \\
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This essentially enables Dahmani and Guirardel to reduce the isomorphism problem for rigid relatively hyperbolic groups (with r.f. parabolics, and some other technical criteria) to the isomorphism problem for hyperbolic groups *with* torsion.

Finding the JSJ

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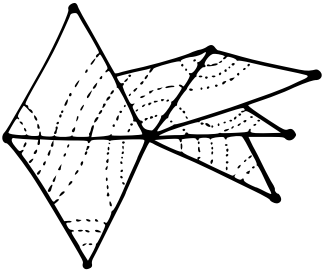
With Nicholas' track finding algorithm* (arXiv 2011)



it is possible to decide whether a torsion free relatively hyperbolic group $(G, \mathcal{P})$ admits an essential elementary splitting without having to resort to solving equations, provided $\mathcal{P}$ belongs to a class of *algorithmically tractable groups.*

Finding the JSJ

Following Guirardel and Levitt we chose our canonical JSJ decomposition to be dual to a $(G, \mathcal{P})$-tree T^c which is obtained as the *tree of cyclinders for co-elementarity* of some tree $(G, \mathcal{P})$-tree T living in the $(G, \mathcal{P})$ JSJ deformation space.

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Theorem (Dahmani-T)

If $(G, \mathcal{P})$ is relatively hyperbolic with $\mathcal{P}$ algorithmically tractable and effectively coherent, then we can compute the canonical JSJ decomposition of $(G, \mathcal{P})$.

Putting things together (general nonsense)

Let $\mathbb{X}$, $\mathbb{X}'$ graphs of groups with underlying bipartite directed graphs X, X'. Let $x \mapsto x'$ be an isomorphism from X to X'. Suppose for each $v \in V(X)$ we have some $\psi_v : \Gamma_{v'}, \xrightarrow{\sim} \Gamma_v$.

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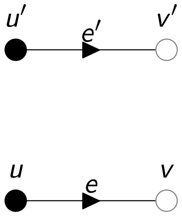

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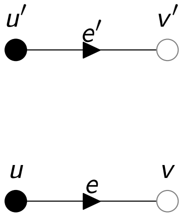

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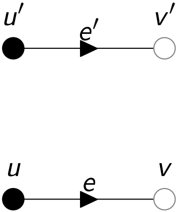

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Then the map $X' \to X$ given by $x' \mapsto x$ is induced by an isomorphism $\pi_1(\mathbb{X}) \xrightarrow{\sim} \pi_1(\mathbb{X}')$ if and only if there exist for each vertex group Γ_w an *extension adjustement* $\alpha_w \in \mathrm{Aut}(\Gamma_w)$ and for each edge e elements $g_{e^-} \in \Gamma_{0(e)}, g_{e^+} \in \Gamma_{t(e)}$. Such that:

$$\begin{array}{ccccc} \Gamma'_u & \xleftarrow{t_{e'}} & \Gamma_{e'} & \xrightarrow{i_{e'}} & \Gamma_{v'} \\ \psi_u \downarrow & & & & \downarrow \psi_v \\ \Gamma_u & & & & \Gamma_v \end{array}$$

$$\Gamma_u \xleftarrow{t_e} \Gamma_e \xrightarrow{i_e} \Gamma_v$$

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\Gamma'_u & \Gamma_{e'} \arrow[l, hook', "t_{e'}"'] \arrow[r, hook, "i_{e'}"] & \Gamma_{v'} \arrow[d, "\psi_v"] \\
\arrow[from=1-1, d, "\psi_u"'] \Gamma_u \arrow[d, "\alpha_u"'] & & \Gamma_v \arrow[d, "\alpha_v"] \\
\Gamma_u & & \Gamma_v \\
\Gamma_u & \Gamma_e \arrow[l, hook', "t_e"'] \arrow[r, hook, "i_e"] & \Gamma_v
\end{tikzcd}
$$

Putting things together (general nonsense)

Then the map $X' \to X$ given by $x' \mapsto x$ is induced by an isomorphism $\pi_1(\mathbb{X}) \xrightarrow{\sim} \pi_1(\mathbb{X}')$ if and only if there exist for each vertex group Γ_w an *extension adjustement* $\alpha_w \in \mathrm{Aut}(\Gamma_w)$ and for each edge e elements $g_{e^-} \in \Gamma_{0(e)}, g_{e^+} \in \Gamma_{t(e)}$. Such that:

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\Gamma_u \arrow[d, "\mathrm{ad}_{g_{e^-}}"'] & & \Gamma_v \arrow[d, "\mathrm{ad}_{g_{e^+}}"] \\
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$$
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 & s \arrow[ddddd, mapsto, bend left=90] & \\
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\Gamma_u \arrow[d, "\alpha_u"'] & & \Gamma_v \arrow[d, "\alpha_v"] \\
\Gamma_u \arrow[d, "\mathrm{ad}_{g_{e^-}}"'] & & \Gamma_v \arrow[d, "\mathrm{ad}_{g_{e^+}}"] \\
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\Gamma_u \arrow[d, "\alpha_u"'] & & \Gamma_v \arrow[d, "\alpha_v"] \\
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 & s'' \qquad s' &
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$$
\begin{tikzcd}
 & s \arrow[dddddddd, mapsto, bend right=80] \arrow[dddddddd, mapsto, bend left=80] & \\
\Gamma'_u \arrow[d, "\psi_u"'] & \Gamma_{e'} \arrow[l, hook', "t_{e'}"'] \arrow[r, hook, "i_{e'}"] & \Gamma_{v'} \arrow[d, "\psi_v"] \\
\Gamma_u \arrow[d, "\alpha_u"'] & & \Gamma_v \arrow[d, "\alpha_v"] \\
\Gamma_u \arrow[d, "\mathrm{ad}_{g_{e^-}}"'] & & \Gamma_v \arrow[d, "\mathrm{ad}_{g_{e^+}}"] \\
\Gamma_u & \Gamma_e \arrow[l, hook', "t_e"'] \arrow[r, hook, "i_e"] & \Gamma_v \\
 & & \\
 & & \\
 & & \\
 & s'' = s' &
\end{tikzcd}
$$

Putting things together: outer automorphisms

Let $(G, \mathcal{P})$ relatively hyperbolic with $\mathcal{P} = [P_1] \cup \cdots \cup [P_s]$, clasically we have $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$

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Fact 1: For each P_i (since these are self-normalized) there is a well defined "restriction"

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Fact 1: For each P_i (since these are self-normalized) there is a well defined "restriction"

$$\mathrm{Out}(G, \mathcal{P}) \to \mathrm{Out}(P_i)$$

Fact 2: If $(G, \mathcal{P})$ is *rigid* then $\mathrm{Out}(G, \mathcal{P})$ is *finite*..

Putting things together: the relatively hyperbolic case

Each white vertex groups Γ_v comes equipped with an *induced rel. hyp. structure* $(\Gamma_v, \mathcal{P}_v)$ in which the *images of the edge groups are parabolic subgroups.** and such that $(\Gamma_v, \mathcal{P}_v)$ is *rigid*.

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$$
\begin{tikzcd}
 & S \arrow[dddddl, mapsto] & \\
\Gamma'_u \arrow[d, "\psi_u"'] & \Gamma_{e'} \arrow[l, hook', "t_{e'}"'] \arrow[r, hook, "i_{e'}"] & \Gamma_{v'} \arrow[d, "\psi_v"] \\
\Gamma_u & & \Gamma_v \arrow[d, "\alpha_v"] \\
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S'_{\alpha_v, g_{e^+}} & &
\end{tikzcd}
$$

Take on only finitely many possible values up to conjugacy in Γ_e.

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T & \Gamma_u & & \Gamma_v \arrow[d, "\alpha_v"] \\
 & & & \Gamma_v \arrow[d, "\mathrm{ad}_{g_{e^+}}"] \\
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$$
\begin{tikzcd}
 & & S \arrow[ddll, mapsto] \arrow[ddddddl, mapsto] & \\
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T & \Gamma_u & & \Gamma_v \arrow[d, "\alpha_v"] \\
 & & & \Gamma_v \arrow[d, "\mathrm{ad}_{g_{e^+}}"] \\
 & \Gamma_u & \Gamma_e \arrow[l, hook', "t_e"'] \arrow[r, hook, "i_e"] & \Gamma_v \\
 & S' & &
\end{tikzcd}
$$

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So let T be defined as:

$$
\begin{tikzcd}
 & & S \arrow[ddll, mapsto] \arrow[ddddd, mapsto, bend left=60] & \\
 & \Gamma'_u \arrow[d, "\psi_u"'] & \Gamma_{e'} \arrow[l, hook', "t_{e'}"'] \arrow[r, hook, "i_{e'}"] & \Gamma_{v'} \arrow[d, "\psi_v"] \\
T & \Gamma_u \arrow[d, "\alpha_{T,S'}"'] & & \Gamma_v \arrow[d, "\alpha_v"] \\
 & \Gamma_u \arrow[d, "\mathrm{ad}_{g_{T,S'}}"'] & & \Gamma_v \arrow[d, "\mathrm{ad}_{g_{e^+}}"] \\
 & \Gamma_u & \Gamma_e \arrow[l, hook', "t_e"'] \arrow[r, hook, "i_e"] & \Gamma_v \\
 & S' & &
\end{tikzcd}
$$

We need to find $\alpha_{T,S'}, g_{T,S'}$ so that $ad_{G_{T,S'}} \circ \alpha_{T,S'}(T) = S'$.

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$$
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\end{tikzcd}
$$

We need to find $\alpha_{T,S'}, g_{T,S'}$ so that $ad_{G_{T,S'}} \circ \alpha_{T,S'}(T) = S'$. Things get more complicated because black vertex groups can have multiple incident edge groups.*.

Putting things together: the mixed Whitehead problem

Let $(S_1, \ldots, S_n)$, $(T_1, \ldots, T_n)$ be a tuple of tuples of elements in G the *mixed Whitehead problem* (MWHP) asks whether there exists some $\alpha \in \mathrm{Aut}(G)$ and $g_1, \ldots, g_n \in G$ such that $g_i^{-1}\alpha(S_i)g_i = T_i$.

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The MWHP is solvable in the class of f.g. nilpotent groups.

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Bogopolski and Ventura also proved the MWHP (and coined the term) for t.f. hyperbolic groups.

Putting things together: reduction to the mixed Whitehead problem

Suppose it was possible for every white vertex v group to *construct* the finite list of images $S'_{\alpha_v, g_{e^+}} = g_{e^+}^{-1} \alpha_v \circ \psi_v(S) g_{e^+}$ *up to conjugacy in* Γ_u.

$$
\begin{tikzcd}
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$$
\begin{tikzcd}
 & S \arrow[dddddl, mapsto, rounded corners, to path={-| ([xshift=4em]\tikztotarget.east |- \tikztostart.east) |- (\tikztotarget)}] & \\
\Gamma'_u \arrow[d, "\psi_u"'] & \Gamma_{e'} \arrow[l, hook', "t_{e'}"'] \arrow[r, hook, "i_{e'}"] & \Gamma_{v'} \arrow[d, "\psi_v"] \\
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\end{tikzcd}
$$

Then the isomorphism problem can be reduced to finitely many instances of the MWHP in the black vertex groups.

Putting things together: orbit computations

Let $(G, \mathcal{P})$ with $\mathcal{P} = [P_1] \cup \cdots \cup [P_s]$ be relatively hyperbolic, the we have the well defined "restriction" map $\mathrm{Out}(G, \mathcal{P}) \to \mathrm{Out}(P_i)$.

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$$r : \mathrm{Out}(G, \mathcal{P}) \to \bigoplus_{i=1}^{n} \mathrm{Out}(P_i)$$

is finite.

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For our purposes it is in fact sufficient to compute T, i.e. find a set $\mathcal{L} = \{\alpha_, \ldots, \alpha_r\} \subset \mathrm{Aut}(G, \mathcal{P})$ which give representatives of T, to construct the finite list of images needed for our reduction to the MWHP.

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By the way we do not know how to compute $Out(G, \mathcal{P})$ (all known methods involve equationnal methods.)

A digression:separating torsion in $\mathrm{Out}(P)$

Let P be a group, we say that *congruences of P separate the torsion in $Out(P)$* if there is some finite index characteristic subgroup $P_0 \lhd P$ such that in the natural map

$$\pi_{P_0} : \mathrm{Out}(P) \to \mathrm{Out}(P/P_o)$$

every finite order $[\alpha] \in Out(P)$ survives, equivalently the kernel of π_{P_0} is *torsion free.*

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every finite order $[\alpha] \in Out(P)$ survives, equivalently the kernel of π_{P_0} is *torsion free.*
A celebrated example is for $P = \mathbb{Z}^m$, then every finite order element of $GL(m, \mathbb{Z})$ survives in $GL(m, \mathbb{Z}/n\mathbb{Z})$ for n sufficiently large.

A digression:separating torsion in $\mathrm{Out}(P)$

We say that *congruences of P effectively separate the torsion in $Out(P)$* if there is an algorithm which finds a finite index characteristic subgroup P_0 which is "deep enough" so that $\ker(\pi_{P_0})$ is torsion free.

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Theorem (Segal, private communication)

If P polycyclic-by-finite then congruences of P separate the torsion in $Out(P)$.

Theorem (Dahmani-T)

There is a uniform algorithm for all f.g. nilpotent groups so that if N is nilpotent then congruences of N effectively separate the torsion in $\mathrm{Out}(N)$.

Putting things together: orbit computations

So going back to our $(G, \mathcal{P})$ with $\mathcal{P} = [P_1] \cup \cdots \cup [P_s]$ if congruences in P_i effectively separate the torsion in $\mathrm{Out}(P_i)$ then we can pick N so large that for $n \geq N$,

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- The n^{th} characteristic Dehn filling $(G/K_n, \mathcal{P}_n)$ is relatively hyperbolic with $\mathcal{P}_n = [P_1/K_n] \cup \cdots \cup [P_s/K_n]$.
- The map

$$T \to \bigoplus_{i=1}^{n} \mathrm{Out}(P_i/K_n)$$

induced by

$$\begin{array}{ccc} \mathrm{Out}(G,\mathcal{P}) & \longrightarrow & \mathrm{Out}(P_i) \\ \downarrow & & \downarrow \\ \mathrm{Out}(G/K_n,\mathcal{P}_n) & \longrightarrow & \mathrm{Out}(P_i/K_n) \end{array}$$

is *injective*.

Computing T

The computability of T now follows from an enumeration argument that uses the Dahmani-Guirardel isomorphism lifting principle,

Computing T

The computability of T now follows from an enumeration argument that uses the Dahmani-Guirardel isomorphism lifting principle,i.e.

$$\begin{tikzcd}
(G,\mathcal{P}) \arrow[r, dashed, "\exists\alpha", "\approx"'] \arrow[d, two heads] & (H,\mathcal{Q}) \arrow[d, two heads] \\
(G/K_m,\mathcal{P}_m) \arrow[r, "\alpha_m", "\approx"'] \arrow[d, two heads] & (H/L_m,\mathcal{Q}_m) \arrow[d, two heads] \\
(G/K_n,\mathcal{P}_n) \arrow[r, "\alpha_n", "\approx"'] & (H/L_n,\mathcal{Q}_n)
\end{tikzcd}$$

and the fact that automorphisms of a group with solvable word problem are enumerable.

The criteria

A class $\mathcal{C}$ of groups is said to be *algorithmically tractable* if

- The finite presentation of the groups in $\mathcal{C}$ are recursively enumerable.

Limit groups, certain classes of small cancellation groups (i.e. those with some type of effective coherence), and polycyclic-by-finite groups lie in this class.

The criteria

A class $\mathcal{C}$ of groups is said to be *algorithmically tractable* if

- The finite presentation of the groups in $\mathcal{C}$ are recursively enumerable.
- The conjugacy problem is uniformly solvable in $\mathcal{C}$.

Limit groups, certain classes of small cancellation groups (i.e. those with some type of effective coherence), and polycyclic-by-finite groups lie in this class.

The criteria

A class $\mathcal{C}$ of groups is said to be *algorithmically tractable* if

- The finite presentation of the groups in $\mathcal{C}$ are recursively enumerable.
- The conjugacy problem is uniformly solvable in $\mathcal{C}$.
- The generation problem is uniformly solvable in $\mathcal{C}$.

Limit groups, certain classes of small cancellation groups (i.e. those with some type of effective coherence), and polycyclic-by-finite groups lie in this class.

Theorem (Dahmani-T)

There is an algorithm which takes explicit presentations of torsion free relatively hyperbolic groups $(G, [P_1] \cup \cdots \cup [P_s])$, $(H, [Q_1] \cup \cdots \cup [Q_s])$ *and provided the* P_i, Q_j *lie in a class* $\mathcal{C}$ *of groups*

1. *that is algorithmically tractable,*
2. *that is uniformly effectively coherent,*
3. *in which we can solve the isomorphism problem,*
4. *in which the mixed Whitehead problem is uniformly solvable, and*
5. *in which congruence effectively separate torsion;*

then the algorithm decides whether or not the two groups

$$(G, [P_1] \cup \cdots \cup [P_s]), (H, [Q_1] \cup \cdots \cup [Q_s])$$

are isomorphic.