Quasi-isometrically rigid graphs of surface groups

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yGagta 2017, Bilbao

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According to Cornelia Drutu and Misha Kapovich, at least according to their manuscript *Geometric Group Theory*, the fundamental questions are:

Question

If G and G' are quasiisometric groups, to what extent to G and G' share the same algebraic properties?

Question

If a group G is quasiisometric to a metric space X, what geometric properties (or structures) on X translate to interesting properties of G

Given G one can always obtain a quasi-isometric G' by either attaching a finite subgroup, e.g. $G' = G \oplus F$, or passing to a finite index subgroup.

We say that G and G' are (abstractly) *commensurable* if there are finite subgroups $F \triangleleft G, F \triangleleft G'$ finite such that there are isomorphic finite index subgroups

$$G/F \overset{\text{f.i.}}{\geq} H \approx H' \overset{\text{f.i.}}{\leq} G'/F'.$$

Note: if G, G' are virtually torsion-free (e.g. residually finite hyperbolic) we can ignore the finite groups F, F'.

A *class* $\mathcal{X}$ *of groups* is *quasi-isometrically rigid* if any group quasi-isometric to some $G \in \mathcal{X}$ is in $\mathcal{X}$.

A *group* G is *quasi-isometrically rigid (q.i. rigid)* if

$$H \text{ q.i. to } G \Rightarrow H \text{ commensurable with } G.$$

Generally quasi-isometry classes of groups split into finer commensurability classes.

From R.J. Spatzier's *An Invitation to Rigidity Theory* the following combination of the works of Casson, Chow, Drutu, Eskin, Farb, Gabai, Gromov, Jungreis, Kleiner, Leep, Pansu, Schwartz, Sullivan and Tukia, which itself builds on decades of (Lie group) rigidity theory gives starting with work of Selberg, Mostow, and other famous people:

Theorem

If a finitely generated group Γ *is quasi-isometric to an irreducible lattice in a semisimple Lie group* G*, then it is commensurable with a (possibly different) lattice in* G*.*

Gromov's celebrated polynomial growth theorem implies that the class of finitely generated nilpotent groups is q.i. rigid. f.g. nilpotent groups have a finite index torsion-free subgroup. If N is a torsion-free f.g. nilpotent group then its Mal'cev $\mathbb{Q}$-completion $N^{\mathbb{Q}} \leq UT(n_N, \mathbb{Q})$ classifies its commensurability class.

If N, M are t.f.f.g. nilpotent groups with isometric Mal'cev $\mathbb{R}$-completions $N^{\mathbb{R}} \approx M^{\mathbb{R}}$ then they are q.i. but it is known that

$$N^{\mathbb{R}} \approx M^{\mathbb{R}} \not\Rightarrow N^{\mathbb{Q}} \approx M^{\mathbb{Q}};$$

so nilpotent groups are not q.i. rigid in general. Some subclasses, however, such as $\{\mathbb{Z}^n\}$ are rigid.

In the transition from nilpotent to solvable we have:

Theorem (Farb-Mosher)

If $n \geq 2$ then every $\mathrm{BS}(1, n)$ is q.i. rigid.

... so the the commensurability classification within a q.i. class of groups is an active field of investigation.

Tomorrow Alexander Zakharov will discuss the commensurability classification of a q.i. rigid class within the class of partially commutative groups (or raags).

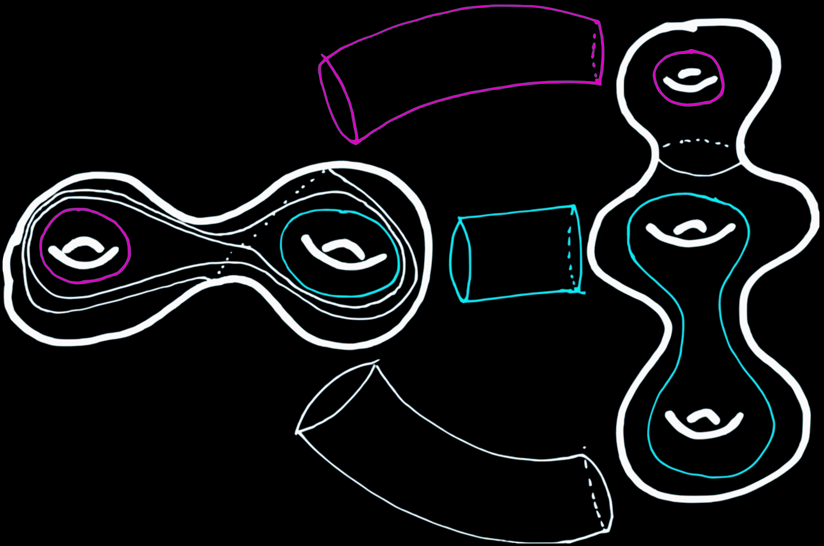
By definition, the class of hyperbolic groups is q.i. rigid. Q.i. hyperbolic groups have homeomorphic boundaries. The converse is not true, but well understood by work of Paulin. Here is a selection of results.

- f.g. free groups (Stallings). One commensurability class. Boundary is a Cantor set.
- Closed surface groups (Gabai, Casson-Jungreis). One commensurability class. Boundary is a circle.
- Convex-cocompact Kleinian groups, i.e. the fundamental group of a hyperbolic 3-manifold possibly with essential boundary (Hassinsky). Many commensurability classes.

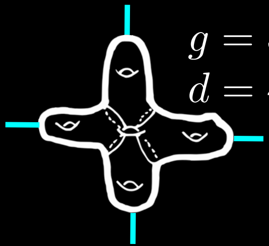
Theorem (Hassinsky)

Let G be hyperbolic and assume ∂G doesn't contain a subset homeomorphic to a Sierpinski carpet. Then G is virtually a convex-cocompact Kleinian group.

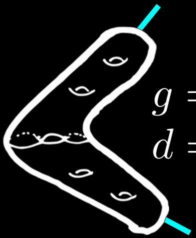
In fact these convex-cocompact Kleinian groups are rather specific: the antepenultimate terms in the cyclic Haken hierarchy are geometric amalgams of surface or free (handlebody groups.)



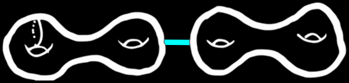
Whyte's example:

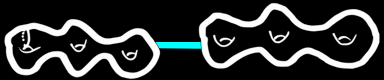


$$g = 5$$
$$d = 4$$

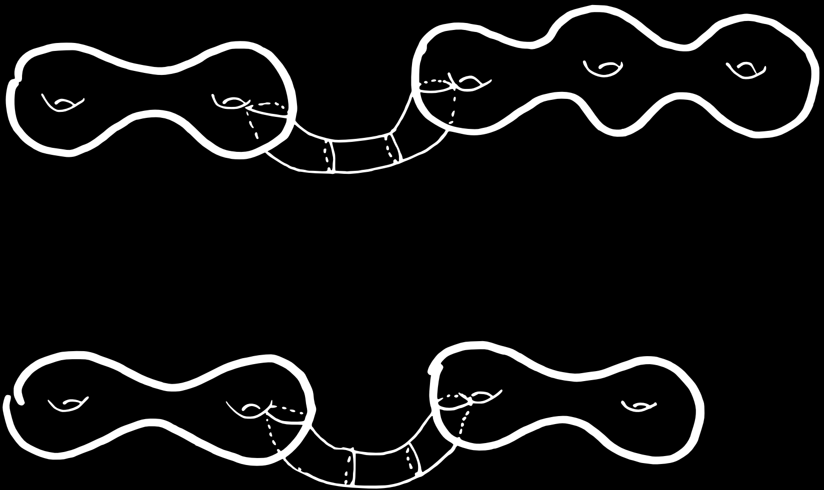


$$g = 5$$
$$d = 2$$





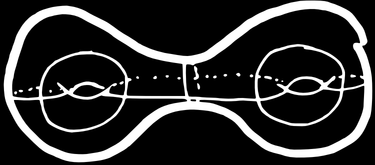
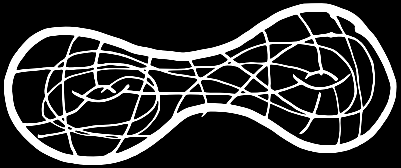
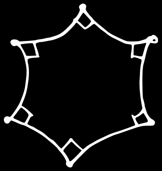
Stark thesis:



Q.i. but not commensurable.

Recent and substantial progress by Pallavi Dani, Emily Stark, and Anne Thomas has been made towards the commensurability classification of geometric amalgams of free groups.

A collection $\{\gamma_i\}$ of geodesic closed curves in a closed hyperbolic surface Σ is *filling* if every component of $\Sigma \setminus (\bigcup \gamma_i)$ is a simply connected polygon.

Theorem (Taam-T)

Let X be obtained by taking a finite collection of closed surfaces and attaching them together by cylinders. If the attaching maps of the cylinders form a filling collection of closed curves in each surface and if $\pi_1(X)$ is hyperbolic, then it is q.i. rigid.

Theorem (Taam-T)

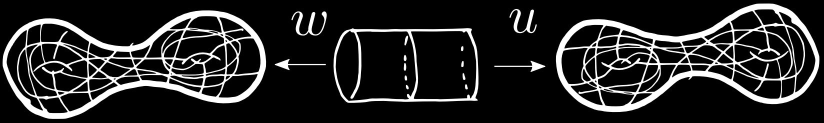
If Γ is a hyperbolic groups whose JSJ decomposition has only rigid-type vertex groups all of which are closed surface groups, then Γ is q.i. rigid.

Corollary

The amalgamated free product

$$\langle a,b,c,d \mid a^{-1}b^{-1}abc^{-1}d^{-1}cd\rangle *_{w=u} \langle \alpha,\beta,\gamma,\delta \mid \alpha^{-1}\beta^{-1}\alpha\beta\gamma^{-1}\delta^{-1}\gamma\delta\rangle$$

*is probably** *going to be rigid if* $w = w(a,b,c,d)$ *and* $u = u(\alpha,\beta,\gamma,\delta)$ *are long enough and picked at random.*



w
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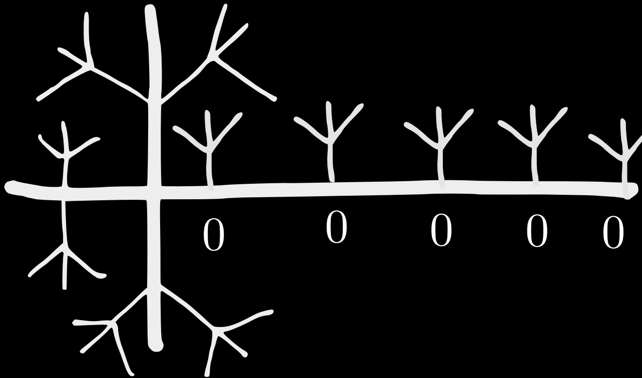
Proposition

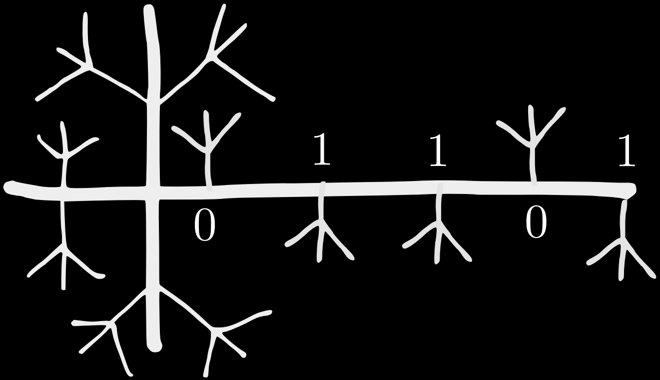
If Γ *is a graph of rigid surface groups (a* **grsg***-group) and* H *is q.i. to* Γ *then both groups virtually act by automorphisms on* $\square^*(\Gamma)$*. In particular, if* $G = \mathrm{Aut}(\square^*(\Gamma))$*, then*

$$\Gamma, H \leq G.$$

Consider the group of automorphisms $G = \mathrm{Aut}(T_4)$ of a 4-regular tree:



Consider the group of automorphisms $G = \mathrm{Aut}(T_4)$ of a 4-regular tree:

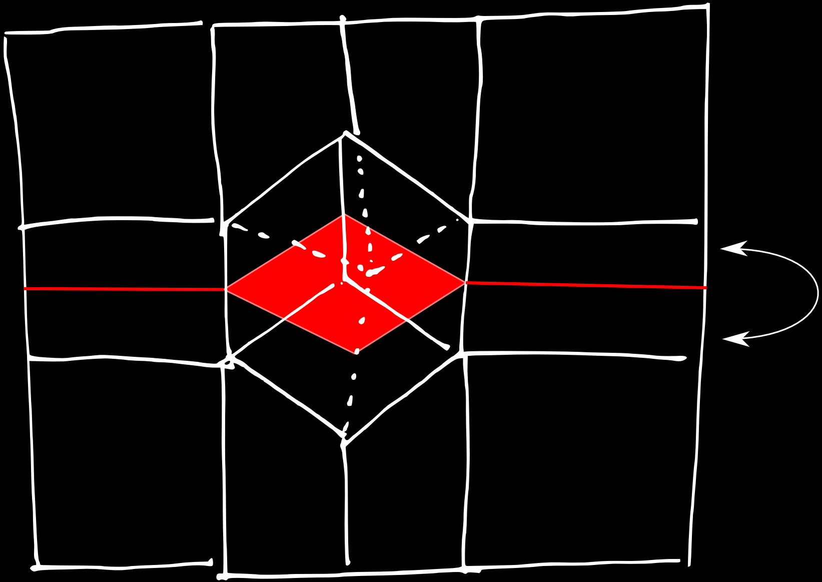


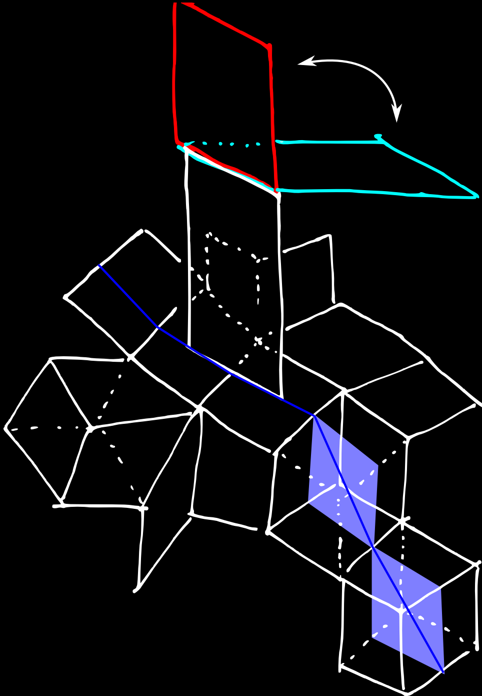
- G is uncountable.
- G is a topological group. An element g is close to 1 if it leaves a large ball centered at 1 fixed and does all its flipping outside. h, k are close if hk^{-1} are close to 1.
- G is a totally disconnected locally compact group. Point stabilizers are profinite (so homeomorphic to Cantor sets) and open in G.

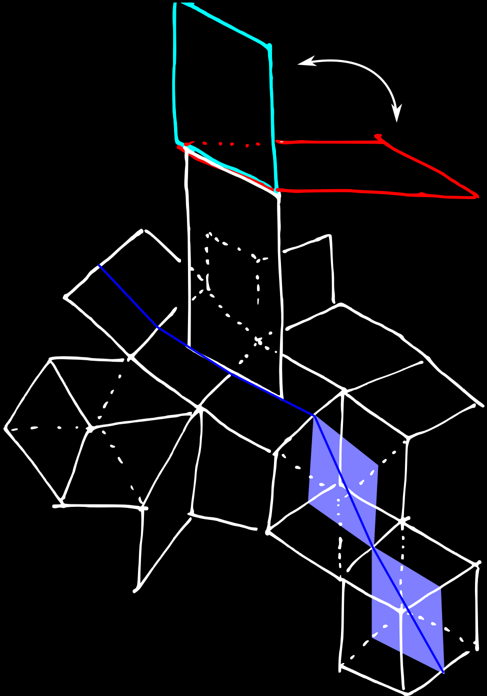
The automorphism group of any reasonable* CW complex Y (e.g. a Cayley complex) is a t.d.l.c. and there is a dichotomy:

- topologically interesting: G is a topological group with cardinality $2^{\aleph_0}$.
- topologically boring: discrete topology. E.g. $Y = \mathrm{Cay}_S(G)$ and maybe $\mathrm{Aut}(Y) = G$. Then G is f.p.

Such an automorphism group will be discrete if the stabilizer of a sufficiently large ball is trivial: the singleton containing the identity is an open set.



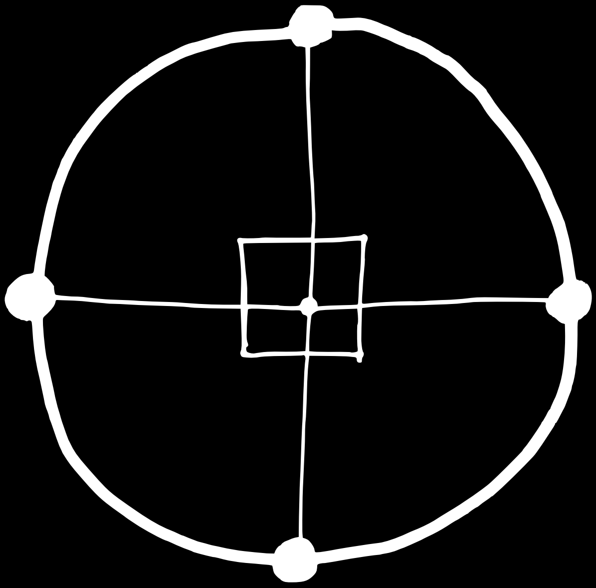




Slogan: being able to flip stuff around independently leads to uncountability and is therefore the enemy of discreteness.

Proposition (Taam-T)

Let $\square$ *be the Sageev cubing of* $(\Sigma, \mathcal{L})$ *a surface equipped with a collection of filling curves, i.e. a waffle.* $\mathrm{Aut}(\square)$ *is discrete if and only if* $\mathcal{L}$ *is filling.*



Proposition

If $\square^*(\Gamma)$ *doesn't admit churro flips or waffle reflections (it is asymmetrical) then* Γ *is q.i. rigid.*

Proof.

$G = \mathrm{Aut}(\square^*(\Gamma))$ is discrete, which means that it is f.g. and acts properly discontinuously and cocompactly (geometrically) on $\square^*(\Gamma)$. Since (passing to f.i. subgroups, if necessary) Γ and H also act geometrically on $\square^*(\Gamma)$ and are subgroups of G they are both finite index in G. Since everything is really crowded and stuff we have

$$[\Gamma : H \cap \Gamma], [H : H \cap \Gamma] \leq [G : \Gamma][G : H].$$

□

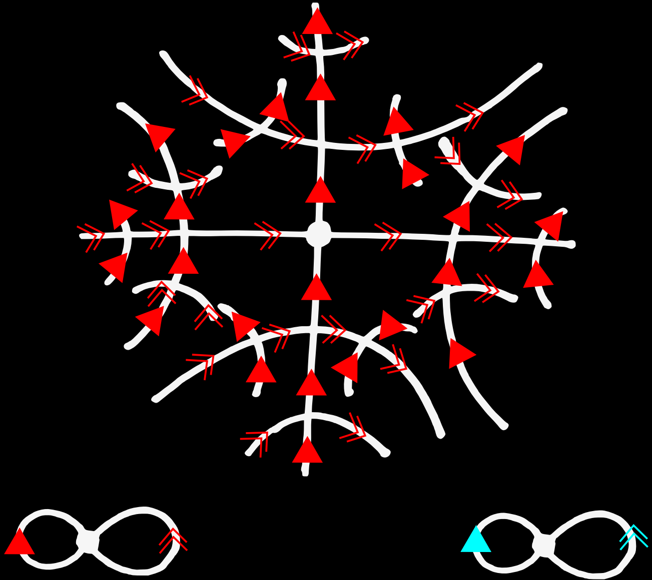
Conjecture (Probably true)

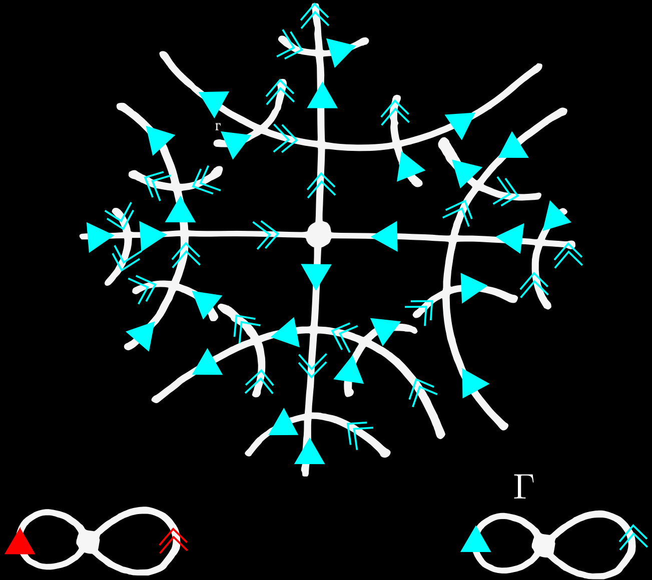
Given a fixed graph with a fixed collection of closed surfaces. A random **grsg***-group obtained by randomly picking words defining the incident cyclic edge groups will probably be q.i. rigid because the churro waffle space will be asymmetrical.*

There therefore (should be) an abundance of such groups.

If $G = \mathrm{Aut}(\square^*(\Gamma))$ is topologically interesting, although H, Γ act cocompactly on $\square^*(\Gamma)$ it is unlikely that they even have non-trivial intersection.

Although we moved away from linear groups, it turns out that the correct perspective is actually to view G as a topological group and to view Γ and H as uniform lattices.

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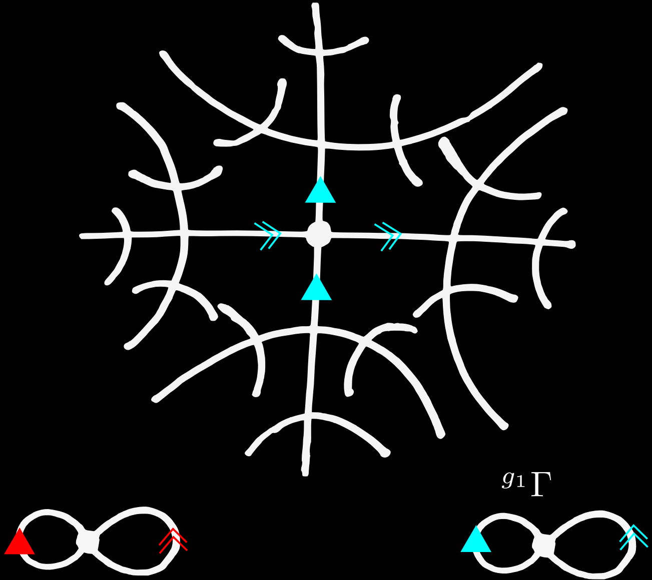
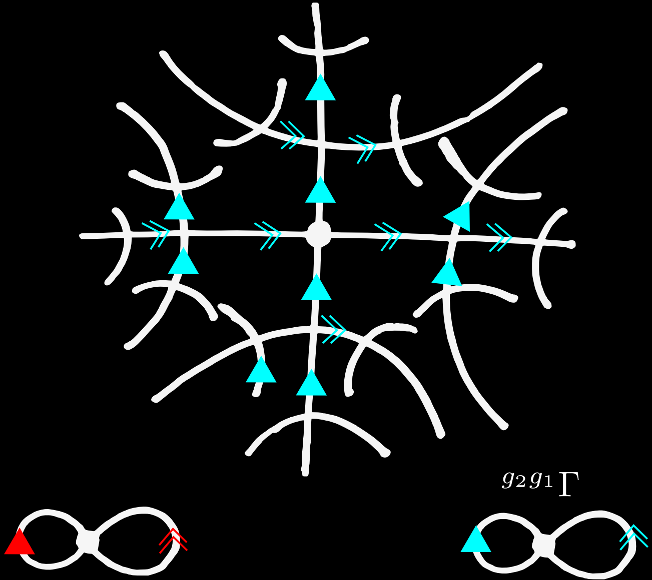
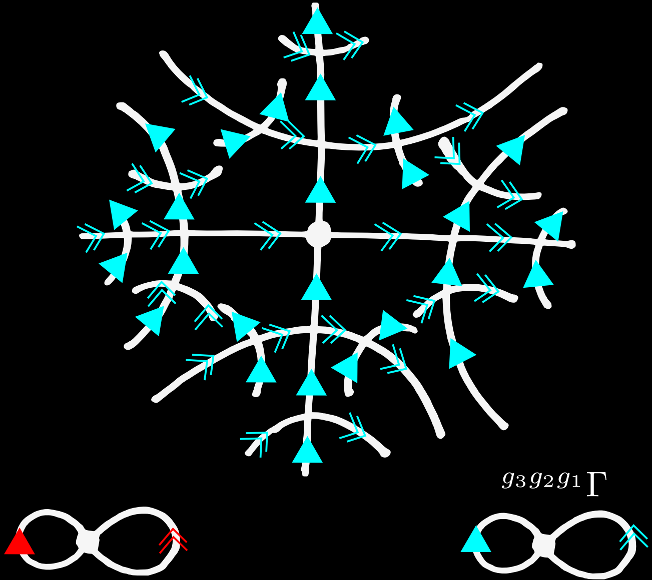
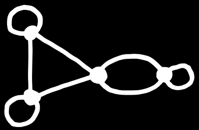

$g_1 \Gamma$

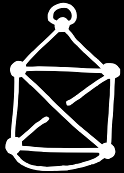


$g_2 g_1 \Gamma$


$g_3 g_2 g_1 \Gamma$

The infinite product $\dots g_3 g_2 g_1$ actually converges in the topology on $G = \mathrm{Aut}(T_4)$ giving $g \Gamma g^{-}1 =_G H$.





Theorem (Leighton)

Let X, Y be finite graphs with the same universal cover, then they have a common finite cover Z.

In particular this produces a common finite index subgroup $\pi_1(Z)$ of $\pi_1(X)$ and $\pi_1(Y)$.

Let $G \circlearrowright X$ and consider the quotient $X \twoheadrightarrow G\backslash X$. We say Δ is a *grouping* of $X \twoheadrightarrow G\backslash X$ if the quotient map $x \mapsto \Delta \cdot x$ realizes the quotient $X \twoheadrightarrow G\backslash X$.

In particular this is a way to define an orbifold structure on $G\backslash X$.

Let T be a well-behaved (e.g. regular) tree.

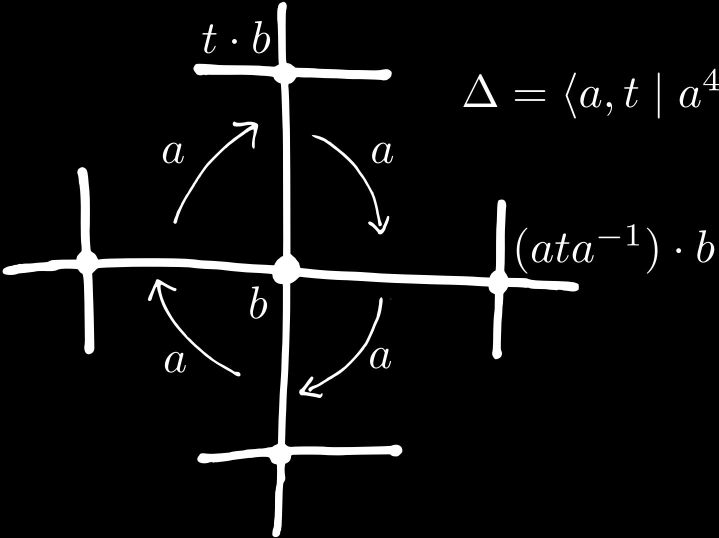
Theorem (Bass, Covering theory for graphs of groups)
Let $G \circlearrowright T$ without inversions and let it be maximal w.r.t groupings of $T \twoheadrightarrow G\backslash T$. Let $\Delta \leq G$ be another grouping of $T \twoheadrightarrow G\backslash T$ (i.e. Δ has the same orbits as G). Let Γ act freely on T, then there is some $g \in G$ such that $\Gamma^g \leq \Delta$.

The proof uses a lot of machinery, but referee of the paper gives a really pleasant 1/2 page self contained proof of this.

Let $G^+ \leq G = \mathrm{Aut}(T_4)$ act without inversions. The quotient is the circle with one vertex. Now consider the HNN extension

$$\Delta = \langle a \mid a^4 \rangle *^t_{\{1\}} = \langle a, t \mid a^4 \rangle \ldots$$

with this action



$$\Delta = \langle a, t \mid a^4 = 1 \rangle$$

This gives a grouping of $T_4 \twoheadrightarrow G^+ \backslash T_4$, but it is a discrete group. It therefore gives a *discrete grouping!*

Discrete groupings are small enough so that cocompact subgroups are finite index and have to have large intersection, but big enough to easily conjugate free lattices into them.

proof of Leightons's Theorem, à la Bass-Kulkarni.

Let Γ, H act freely cocompactly and without inversions on T_4. They sit in G^+. Let Δ be a discrete grouping of $T_4 \twoheadrightarrow G^+ \backslash T_4$. By Bass's conjugacy theorem there is $g, h \in G^+$ such that $\Gamma^g, H^h \overset{\text{f.i.}}{\leq} \Delta$, since they act cocompactly and Δ is discrete, therefore finitely generated. Γ^g, H^h being crowded together must have finite index intersection. $\square$

We were able to adapt the referee's proof of the Bass conjugacy theorem to churro waffle spaces, but then we found this which works great:

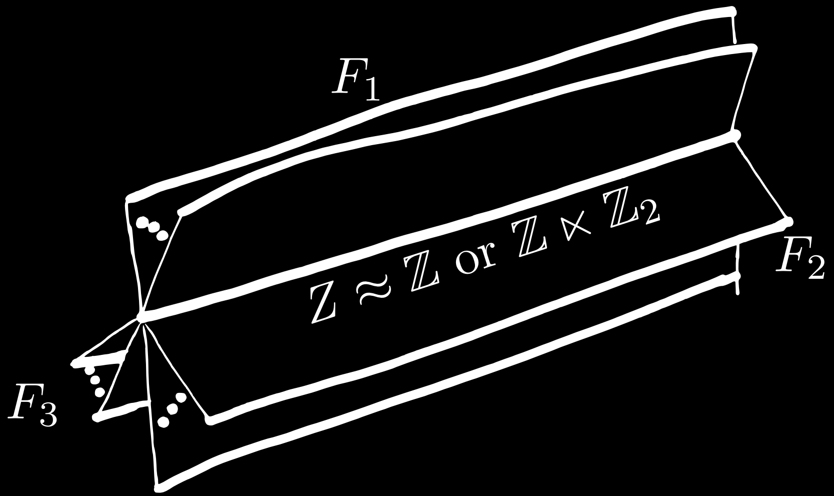
Theorem (Lim-Thomas)

Let $G \circlearrowright X$ be an inversion-free action on a polyhedral complex and let it be maximal w.r.t groupings of $X \twoheadrightarrow G\backslash X$. Let $\Delta \leq G$ be another grouping of $X \twoheadrightarrow G\backslash X$ (i.e. Δ has the same orbits as G). Let Γ act freely on X, then there is some $g \in G$ such that $\Gamma^g \leq \Delta$.

In particular we can use the exact same scheme as for Bass and Kulkarni's proof of Leighton's theorem. All we need are discrete groupings!

We already saw that the isometry groups of waffles are discrete.

For churros we have, e.g.

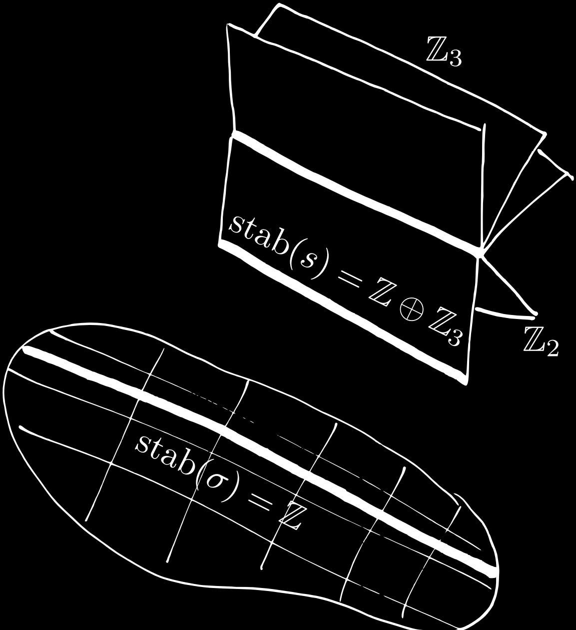


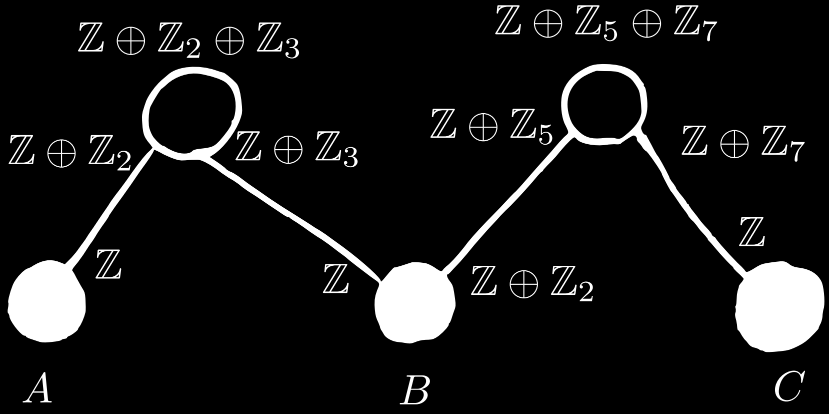
$$\Delta_* = Z \oplus F_1 \oplus F_2 \oplus F_3$$

$\square^*(\Gamma)$ is a tree of spaces so $G\backslash\square^*(\Gamma)$ is a graph of spaces. In fact it is a graph of actions:

- Each vertex comes equipped with a group and a space (a waffle or a churro) and an action on that space.
- Each edge comes equipped with an identification map gluing the edge of a flap to a strand in a churro and an identification of the stabilizers of those spaces.

We want to build a new grouping starting from these discrete discrete actions on individual churros an waffles.


$\mathbb{Z}_3$
$\mathrm{stab}(s) = \mathbb{Z} \oplus \mathbb{Z}_3$
$\mathbb{Z}_2$
$\mathrm{stab}(\sigma) = \mathbb{Z}$



$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$
$\mathbb{Z} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_7$
$\mathbb{Z} \oplus \mathbb{Z}_2$
$\mathbb{Z} \oplus \mathbb{Z}_3$
$\mathbb{Z} \oplus \mathbb{Z}_5$
$\mathbb{Z} \oplus \mathbb{Z}_7$
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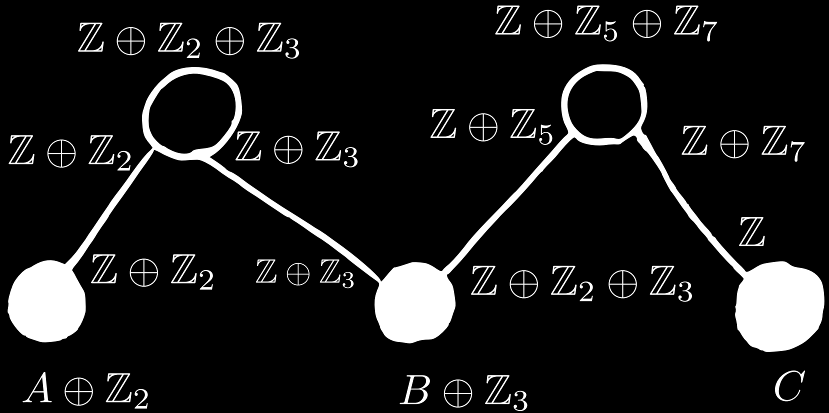
$\mathbb{Z} \oplus \mathbb{Z}_2$
$\mathbb{Z}$
A
B
C



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$\mathbb{Z} \oplus \mathbb{Z}_3$
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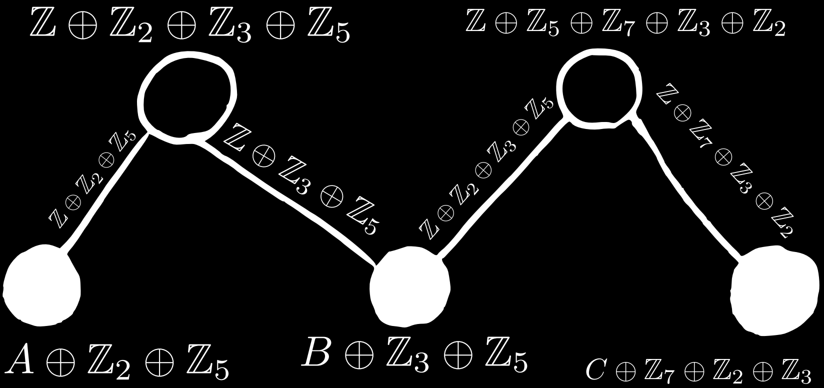
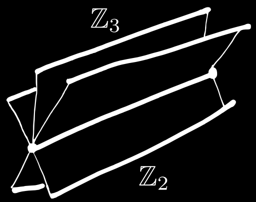


This prevents independent flipping!

Proposition

If Γ is a **grsg***-group whose JSJ has a simply connected underlying graph, then Γ is q.i. rigid.*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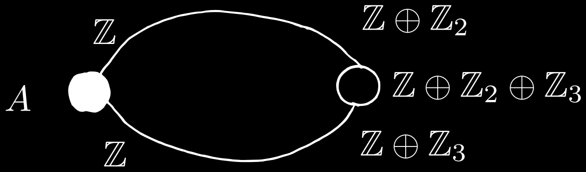
$\mathbb{Z}$
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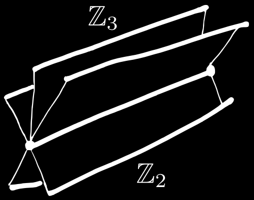
$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$
$\mathbb{Z}$
$\mathbb{Z} \oplus \mathbb{Z}_3$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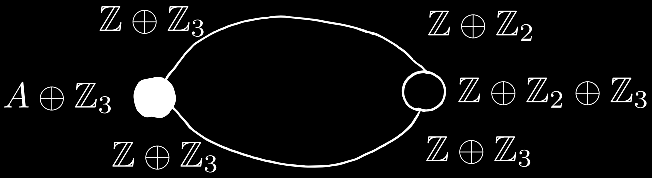
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$A \oplus \mathbb{Z}_3$
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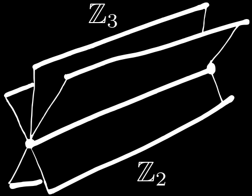
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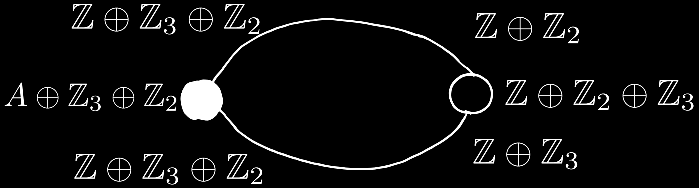
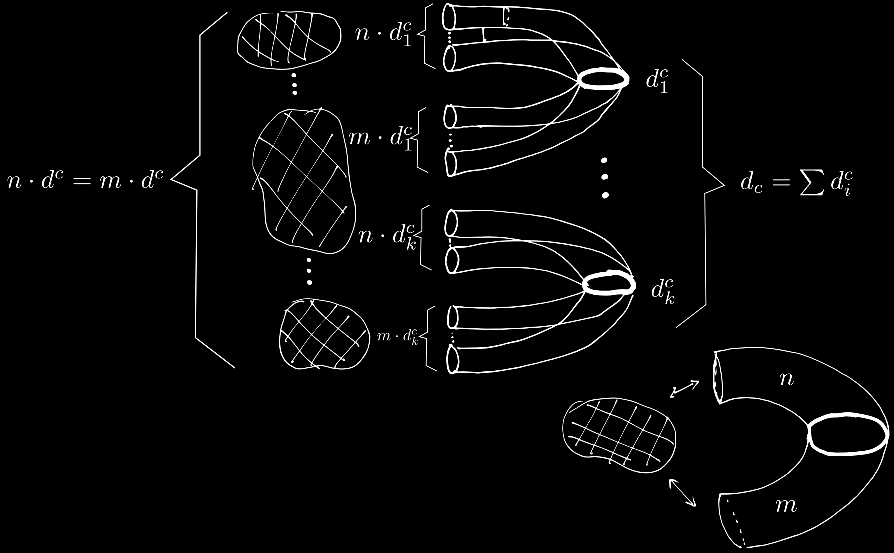
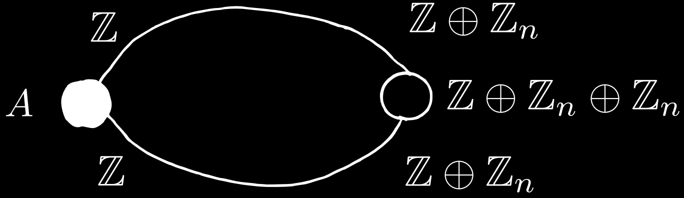

$\mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$
$\mathbb{Z} \oplus \mathbb{Z}_2$
$A \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$
$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$
$\mathbb{Z} \oplus \mathbb{Z}_3$
$\mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$

Once you augment through e the stabilizers for f should already be isomorphic!

Γ acts cleanly on $\square^*(\Gamma)$ if it is torsion-free and if any g that stabilizes a churro stabilizes it pointwise. In particular $G\backslash\square^*(\Gamma)$ must have a *clean and good* orbifold cover. In particular every churro should be adjacent to the full number of flaps.


$n \cdot d^c = m \cdot d^c$
$n \cdot d_1^c$
$m \cdot d_1^c$
$n \cdot d_k^c$
$m \cdot d_k^c$
d_1^c
d_k^c
$d_c = \sum d_i^c$
n
m

In particular $n = m$. So the existence of a clean cover and a degree argument put constraints on the cardinalities of flap families, and therefore on the stabilizers.



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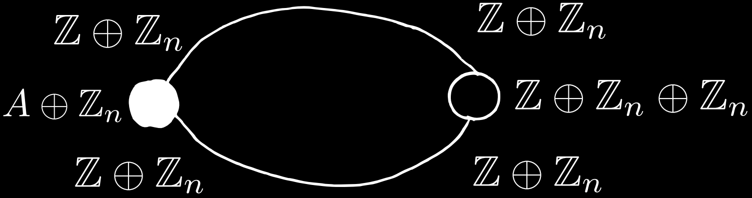

In the general case a detailed examination of orbifold coverings, point preimages, and some basic finite group theory is need to make everything match up.

The isometry groups of churros and waffles still need to be examined closely and produce unexpected difficulties.

Lemma

Let $W = \mathrm{Aut}(\square)$ *be the automorphism group of a waffle. Denote by* W_x *the stabilizer of* x*. Let* $\sigma \subset \square$ *be a strand. Then the function* $x \mapsto |W_x|$ *is either 1 or 2 on some dense open set* $U \subset \sigma$*.*

To prove this we need to use the local properties of CAT(0) geometry, as well as the fact that $\square$ comes from a line pattern in $\mathbb{H}^2$. Here is a calculation in the proof used to contradict that a curve is geodesic:

$$\begin{aligned} \ell(\epsilon) &= \sqrt{\epsilon^2 + L} + \sqrt{(d_2 - \epsilon)^2 + M} \\ \Rightarrow \ell'(\epsilon) &= \frac{\epsilon}{\sqrt{\epsilon^2 + L}} + \frac{\epsilon - d_2}{\sqrt{(d_2 - \epsilon)^2 + M}} \\ \Rightarrow \ell'(0) &= \frac{-d_2}{\sqrt{{d_2}^2 + M}} \neq 0 \Rightarrow \epsilon = 0 \text{ not critical} \end{aligned}$$

Thank you!

Questions:

- Are random cyclic graphs of surface groups q.i.-rigid?
- Is there an analogous result when we replace surface groups with free groups? See Cashen-Macura, Haissinsky-Paoluzzi-Walsh and Dani-Stark-Thomas. It's so hard we could only do it for surface groups!
- Is there a sensible criterion for when a hyperbolic constructible group is q.i. rigid?
- Are there other contexts where cubulations can be used to obtain quasi-isometric rigidity?
- Are churro waffle spaces CAT(0)?
- Do you have a hobby and, if so, what is it?