Limit groups are not freely conjugacy separable

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$\mathbb{F}$– finitely generated free group.

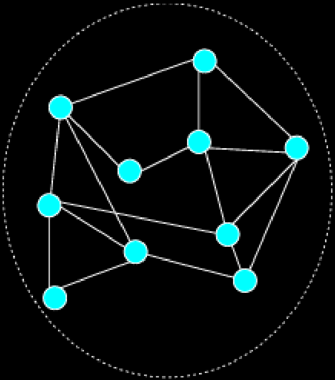
L– limit group $\Leftrightarrow$ L is f.g. and for every $S \subset L, |S| < \infty$ there exists a homomorphism $\varphi_S : L \to \mathbb{F}$ such that $\varphi_S|_S$ is injective.

This property is also called *fully residually free.*

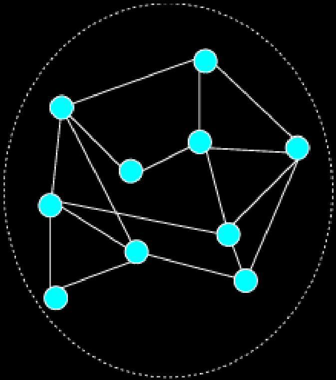
Let $\mathbb{F}_m$ be a free group of rank m and let $X_i \subset \mathbb{F}_m$ be generating sets of cardinality $n \geq m$.

Then the sets of Cayley graphs $\mathbf{Cay}\left(\mathbb{F}_m, X_i\right)$ form a metric space where $d(C_1, C_2) \leq e^{-m}$ if both Cayley graphs have the same balls of radius m about the origin.

So we get a sequence of Cayley graphs of free groups that look like L's Cayley graph.



B_n in $\mathbf{Cay}\,(L, X)$



B_n in $\mathbf{Cay}\,(\varphi(L), \varphi(X))$

This "convergence of balls" property implies that limit groups are $\exists$–equivalent to free groups, i.e. $\mathrm{Th}_{\exists}(L) = \mathrm{Th}_{\exists}(\mathbb{F})$. That this exactly characterizes fully residually free groups is due to Remeslennikov and relies on a property known as being *Equationally Noetherian*.

It is in fact a result of Daniyarova, Miasnikov, Remeslennikov, and independently by Ould-Houcine that Γ Equationally Noetherian implies for all G

$$\mathrm{Th}_{\exists}(G) = \mathrm{Th}_{\exists}(*) \Rightarrow G \text{ fully residually } \Gamma.$$

It is not always the case though that if H is existentially equivalent to Γ whether there is a homomorphisms to Γ. Over a beer M.Hull and T. realized that they independently made the following observation: "if Γ has the limit factoring property then Γ is Equationally Noetherian." In other words the existence of such homomorphisms is *equivalent* to being Equationally Noetherian.

Chagas and Zalesski proved that limit groups are conjugacy separable.

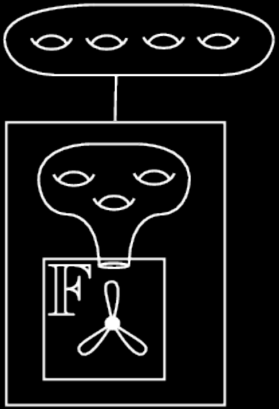
Lioutikova proved that a certain class of groups, iterated centralizer extensions of free groups, are freely conjugacy separable.

$L = \langle x_1, \ldots, x_n \mid r_1(x_1, \ldots, x_n), \ldots r_m(x_1, \ldots . x_n)\rangle$. If L is not freely conjugacy separable then there are elements $g(x_1, \ldots, x_n), h(x_1, \ldots, x_n) \in L$ such that:

$$\mathbb{F} \models \forall x_1, \ldots, x_n \exists t$$
$$\left(\left(\bigwedge_{i=1}^{m} r_i(x_1, \ldots, x_n) = 1 \right) \to \left(t^{-1} g(x_1, \ldots, x_n) t = h(x_1, \ldots, x_n)\right) \right)$$

$\mathbb{F}$


$\mathbb{F}$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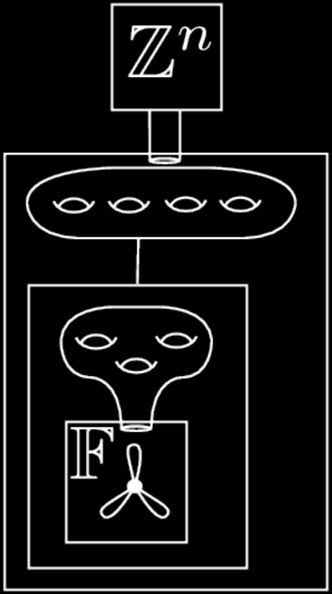
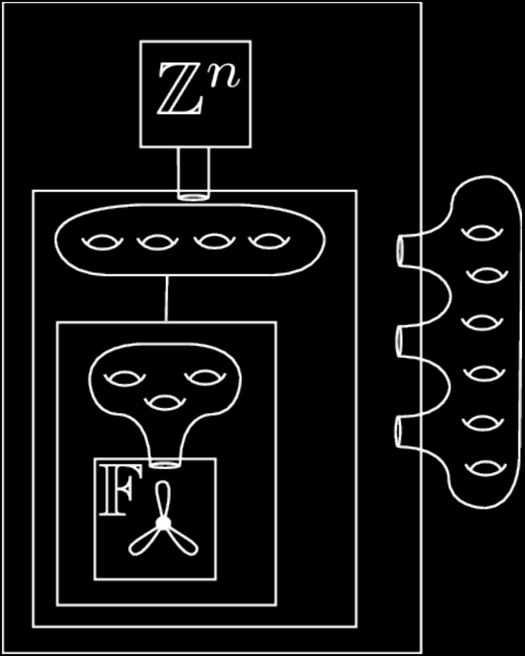
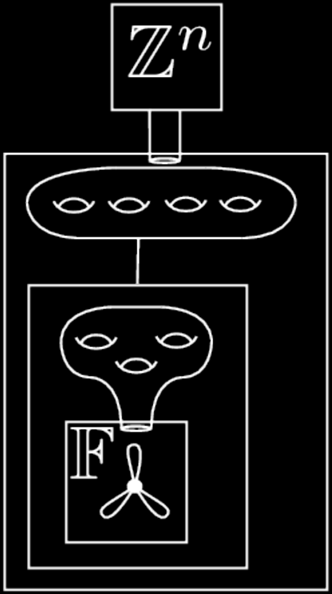

$\mathbb{Z}^n$
$\mathbb{F}$


$\mathbb{Z}^n$
$\mathbb{F}$



$\mathbb{Z}^n$
$\mathbb{F}$


F


$\mathbb{F}$

$\mathbb{F}$

Theorem (Kharlampovich-Myasnikov,Sela)

Let $T = \langle \mathbb{F}, X \mid R(\mathbb{F}, X) \rangle$ be a presentation for a tower based at $\mathbb{F}$. We consider $\mathbb{F}, X \subset T$. Let $S_i(\mathbb{F}, X, Y) = 1$ and $U_j(\mathbb{F}, X, Y) \neq 1$ be finite systems of equations and inequations. Suppose the following held:

$$\mathbb{F} \models \forall X \exists Y \Big(R(\mathbb{F}, X) = 1 \to \bigvee_{i=1}^{m} \big(S_i(\mathbb{F}, X, Y) = 1 \wedge U_i(\mathbb{F}, X, Y) \neq 1 \big) \Big)$$

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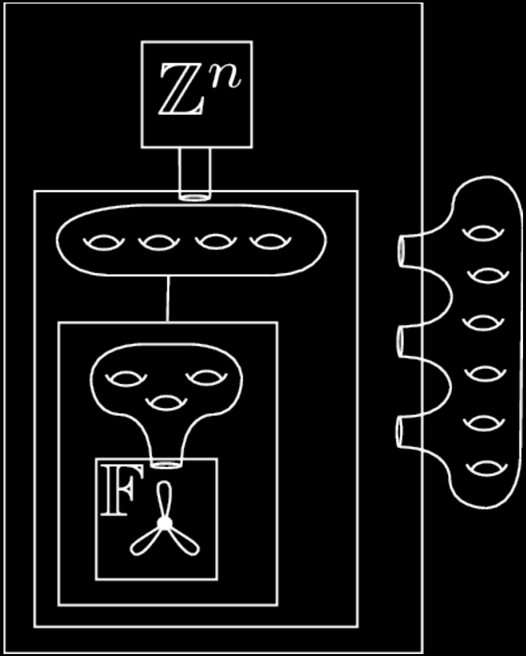
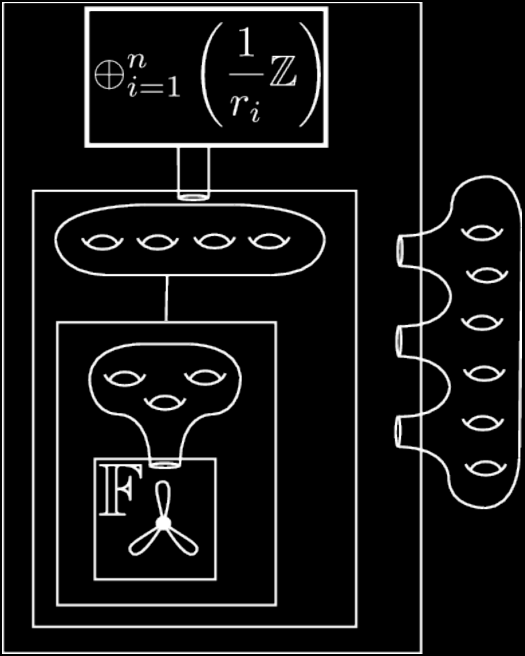
Then there is a closure $T \leq T^*$ *such that*

$$T^* \models \exists Y \bigvee_{i=1}^{m} \big(S_i(\mathbb{F}, X, Y) = 1 \wedge U_i(\mathbb{F}, X, Y) \neq 1 \big)$$

where the elements of the tuple X *and* $\mathbb{F}$ *are interpreted as elements of* T^* *via* $T \leq T^*$.


$\mathbb{Z}^n$
$\mathbb{F}$


$\bigoplus_{i=1}^{n} \left(\frac{1}{r_i} \mathbb{Z} \right)$
$\mathbb{F}$

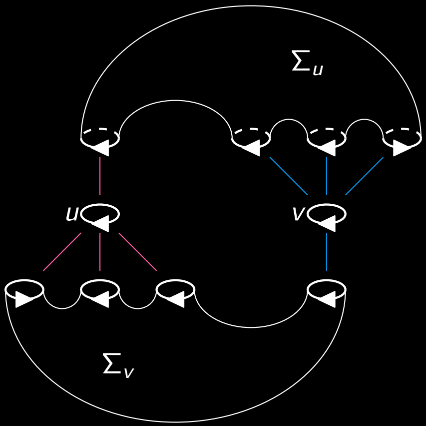
Theorem (Louder-T.)

Towers are freely conjugacy separable.

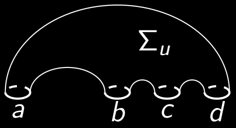
Two elements $u, v \in G$ form a *Magnus pair* if $\langle\langle u \rangle\rangle = \langle\langle v \rangle\rangle$ but u is not conjugate in G to $v^{\pm 1}$.

Theorem (Magnus)

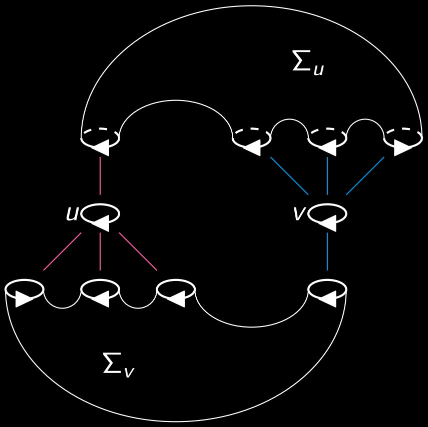
Free groups do not have Magnus pairs, i.e. in $\mathbb{F}$, $\langle\langle u \rangle\rangle = \langle\langle v \rangle\rangle \Rightarrow u$ in conjugate in $\mathbb{F}$ to $v^{\pm 1}$.



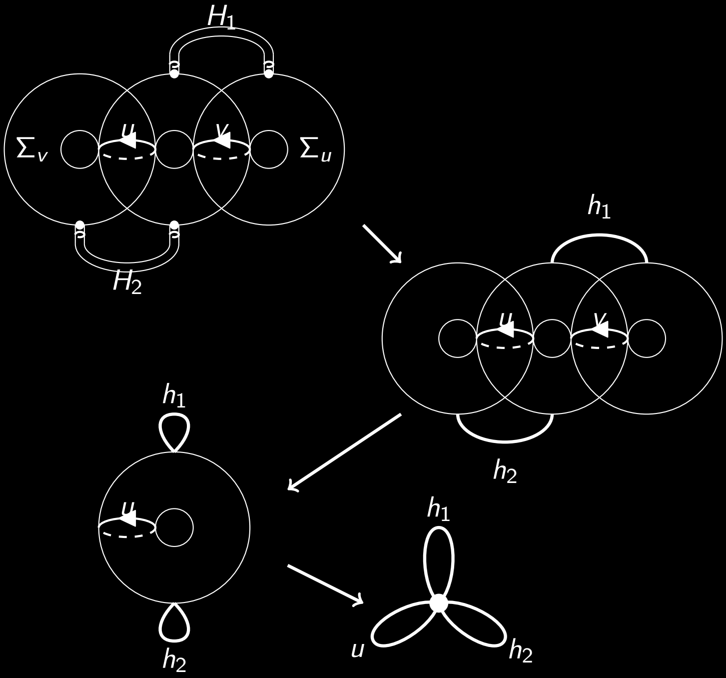
Σ_u
u
v
Σ_v



$$\pi_1(\Sigma_u) = \langle a, b, c, d \mid abcd \rangle$$



Σ_u
u
v
Σ_v



H_1
Σ_v
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v
Σ_u
H_2
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h_2
h_1
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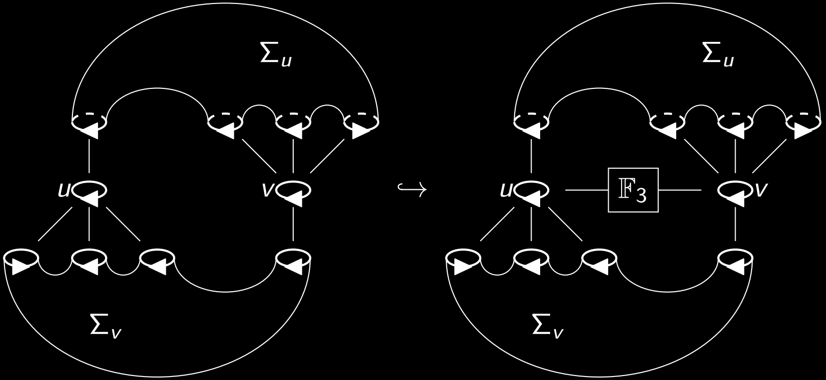
h_2



Σu
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Σv
↪
Σu
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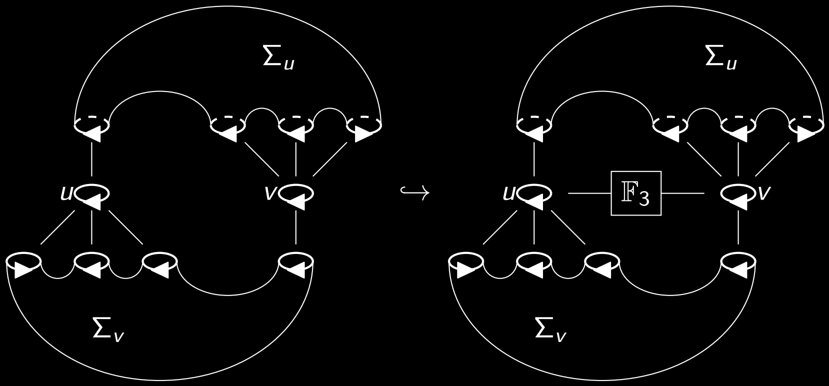
v
Σv



$[u]$ and $[v]$ are the only conjugacy classes that cannot be separated via free quotients.

Theorem (Louder-T.)
Limit groups are not freely conjugacy separable.

A non-trivial word $w(x_1, \ldots, x_n)$ is a C-test word in n letters for $\mathbb{F}_m$ if for any two n-tuples $(A_1, \ldots, A_n), (B_1, \ldots, B_n)$ of elements of $\mathbb{F}_m$ the equality $w(A_1, \ldots, A_n) = w(B_1, \ldots, B_n) \neq 1$ implies the existence of an element $S \in \mathbb{F}_m$ such that $B_i = SA_iS^{-1}$ for all $i = 1, 2, \ldots, n$.

Theorem (S.V. Ivanov, 1998)

For arbitrary $n \geq 2$ there exists a non-trivial word $w_n(x_1, \ldots, x_n)$ which is a C-test word in n letters for any free group $\mathbb{F}_m$ of rank $m \geq 2$. In addition, $w_n(x_1, \ldots, x_n)$ is not a proper power.

Let w be C-test word and consider the double:

$$L = \langle x, y \rangle *_{w(x,y)=w(\xi,\zeta)} \langle \xi, \zeta \rangle.$$

Either the subgroup $\langle x, y \rangle$ has nonabelian image in a free group and any pair of nonconjugate elements of the form $u(x, y), u(\xi, \zeta)$ have conjugate images. Otherwise the subgroups $\langle x, y \rangle$ and $\langle \xi, \zeta \rangle$ have abelian images, in particular elements of their commutator subgroups are sent to the identity.

So there are a lot of conjugacy classes that can't be separated via homomorphisms to free groups.

This group was also independently found by Simon Heil and used to prove that limit groups are not freely subgroup separable.

Questions and answers:

- Does free conjugacy separability isolate the class of towers within limit groups? Not really.
- Do these doubles contain any Magnus pairs? Nope.
- How do Magnus pairs arise in Limit groups? We don't know, but the answer is potentially complicated and uninteresting.
- Why bother? The methods used to solve the Tarski problems for free groups aren't widely understood. Results like these make them more accessible.